① S. s. det. forma generală a șirului dat de recurență

$$x_{n+1} = \frac{n}{n+1} \cdot x_n + \frac{1}{n}, \quad n \geq 1, \quad x_1 = 0$$

Sol: $$x_{n+1} = \frac{n}{n+1} \cdot \left(\frac{n-1}{n} \cdot x_{n-1} + \frac{1}{n-1} \right) + \frac{1}{n}$$

$$= \frac{n-1}{n+1} \cdot x_{n-1} + \frac{n}{(n+1)(n-1)} + \frac{1}{n}$$

$$= \frac{n-1}{n+1} \cdot \left(\frac{n-2}{n-1} \cdot x_{n-2} + \frac{1}{n-2} \right) + \frac{n}{(n+1)(n-1)} + \frac{1}{n}$$

$$= \frac{n-2}{n+1} \cdot x_{n-2} + \frac{1}{n-2} \cdot \frac{n-1}{n+1} + \frac{1}{n-1} \cdot \frac{n}{n+1} + \frac{1}{n}$$

$$= \frac{n-2}{n+1} \cdot x_{n-2} + \frac{1}{n+1} \cdot \left(1 + \frac{1}{n-2} + 1 + \frac{1}{n-1} + 1 + \frac{1}{n} \right)$$

$$= \frac{n-2}{n+1} \cdot x_{n-2} + \frac{1}{n+1} \cdot \left(3 + \left(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} \right) \right)$$

$$= \dots \frac{1}{n+1} \left[n + \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \right]$$

Recurențe liniare de ordin 2 cu coef. constanți

$$x_{n+1} = a x_n + b x_{n-1}, \quad n \geq 1$$

x_0, x_1 dați.

$$x_n = ?$$

$$x_{n+1} - a x_n - b x_{n-1} = 0$$

Dacă am avea $a = (r_1 + r_2)$

$b = -r_1 \cdot r_2$

$$x_{n+1} - r_1 x_n - r_2 x_n + r_1 r_2 x_{n-1} = 0$$

$$(x_{n+1} - r_1 x_n) - r_2 (x_n - r_1 x_{n-1}) = 0$$

$$(x_{n+1} - r_1 x_n) = r_2 (x_n - r_1 x_{n-1})$$

$$= \dots = r_2^n \underbrace{(x_1 - r_1 x_0)}_{\alpha}$$

$$x_{n+1} - r_1 x_n = r_2^n \alpha$$

Am vrea:

$$\left(x_{n+1} - \ldots\right) = r_1 \left(x_n - \ldots\right)$$

$$x_{n+1} - r_2^{n+1}\beta = r_1\left(x_n - r_2^n \beta\right)$$

(clar trb. să apară r_2^n

$$x_{n+1} - r_1 x_n = r_2^n \underbrace{\left(r_2\beta - r_1\beta\right)}_{\alpha}$$

I Dacă $r_2 \neq r_1$ $\quad \beta = \dfrac{\alpha}{r_2 - r_1}$

$$\left(x_{n+1} - r_2^{n+1}\beta\right) = r_1\left(x_n - r_2^n\beta\right)$$

$$= \ldots = r_1^{n+1}\underbrace{\left(x_0 - r_2^0\beta\right)}_{\gamma}$$

$$x_{n+1} = r_2^{n+1}\beta + r_1^{n+1}\gamma$$

$x_n = r_2^n\beta + r_1^n\gamma$ | $x_0 = \beta + \gamma$

$$x_1 = \alpha + r_1 x_0$$
$$= (r_2 - r_1)\beta + r_1(\beta + \gamma)$$
$$= r_2 \beta + r_1 \gamma$$

$$\begin{cases} x_0 = \beta + \gamma \\ x_1 = r_2 \beta + r_1 \gamma \end{cases} \quad r_2 \neq r_1$$

$\Rightarrow$ Se determină β și γ.

II $r_1 = r_2 = r$ Dacă $r = 0 \Rightarrow x_n = 0 \; \forall n \geq 2$

$$x_{n+1} - r x_n = r^n \alpha$$

$$(x_{n+1} - \ldots ?\ldots) = r \cdot (x_n - \ldots ? \ldots)$$

$$\left(x_{n+1} - \beta r^{n+1}(n+1)\right) = r \cdot (x_n - \beta r^n \cdot n)$$

$$x_{n+1} - r \cdot x_n = r^{n+1}\left(\beta(n+1) - \beta \cdot n\right)$$
$$= r^{n+1} \cdot \beta \qquad \beta = \frac{\alpha}{r}$$

$$x_{n+1} - \beta r^{n+1} \cdot (n+1) =$$

$$= r\left(x_n - \beta r^{n} \cdot n\right)$$

$$= \dots = r^{n+1} \cdot \underbrace{x_0}_{\gamma}$$

$$x_{n+1} = r^{n+1} \cdot \gamma + r^{n+1} \cdot (n+1) \cdot \beta$$

$$x_n = r^{n} \cdot \gamma + r^{n} \cdot n \cdot \beta$$

$$x_0 = \gamma$$

$$x_1 = \alpha + r x_0$$

$$x_1 = \beta r + r \gamma$$

$x_1 = r(\beta + \gamma)$ $=,$ Le dit β, γ

$$x_0 = \gamma$$

$$r \neq 0$$

Cum determinăm r_1, r_2 ?

$$x_{n+1} - a x_n - b x_{n-1} = 0$$

$$\Big\downarrow$$

$$x^2 - a x - b = 0$$

Răd. r_1, r_2

$$\begin{cases} r_1 + r_2 = a \\ r_1 \cdot r_2 = -b \end{cases}$$

Concret :

(2) $x_{n+1} = x_n + x_{n-1}$

$$x_0 = x_1 = 1$$

Fibonacci

Folosim idei din demonstrația anterioară:

(3) Termenul general:

$$x_{n+1} = \frac{2}{3} x_n + \left(\frac{1}{2}\right)^n \cdot n, \quad n \geq 0, \; x_0 = 1$$

$$x_{n+1} - \frac{2}{3} x_n = \left(\frac{1}{2}\right)^n \cdot n$$

$$(x_{n+1} - \dots) = \frac{2}{3}(x_n - \dots)$$

Încercăm cu $\dots = \beta \cdot \left(\frac{1}{2}\right)^{n+1} \cdot (n+1)$,

respectiv $\beta \cdot \left(\frac{1}{2}\right)^n \cdot n$

$$\left(x_{n+1} - \beta \cdot \left(\frac{1}{2}\right)^{n+1} \cdot (n+1)\right) = \frac{2}{3}\left(x_n - \beta \cdot \left(\frac{1}{2}\right)^n \cdot n\right)$$

$$x_{n+1} - \frac{2}{3} x_n = \beta \cdot \left(\frac{1}{2}\right)^n \cdot \underbrace{\left(\frac{n+1}{2} - \frac{2}{3} \cdot n\right)}_{\frac{3n+3-4n}{6} = \frac{3-n}{6}}$$

vrem zero $\forall$ n

Clar, punem în loc de

$$\beta \cdot \left(\frac{1}{2}\right)^n \cdot n$$

$$\beta \cdot \left(\frac{1}{2}\right)^n \cdot (n+3)$$

$$\left(x_{n+1} - \beta \cdot \left(\frac{1}{2}\right)^{n+1} (n+4)\right) = \frac{2}{3}\left(x_n - \beta \cdot \left(\frac{1}{2}\right)^n \cdot (n+3)\right)$$

$$x_{n+1} - \frac{2}{3} x_n = \beta \cdot \left(\frac{1}{2}\right)^n \cdot \left(\frac{n+4}{2} - \frac{2}{3} \cdot (n+3)\right)$$

$$= \beta \cdot \left(\frac{1}{2}\right)^n \cdot \frac{3n + \cancel{12} - 4n - \cancel{12}}{6}$$

$$\beta = -6$$

$$\left(x_{n+1} + 3\left(\frac{1}{2}\right)^n \cdot (n+4)\right) = \frac{2}{3} \cdot \left(x_n + 3 \cdot \left(\frac{1}{2}\right)^{n-1} \cdot (n+3)\right)$$

$$= \dots =$$

$$= \left(\frac{2}{3}\right)^{n+1} \cdot \left(\underset{\substack{\parallel \\ 1}}{x_0} + 3 \cdot 2 \cdot 3\right)$$

$$= \left(\frac{2}{3}\right)^{n+1} \cdot 19$$

$$x_{n+1} = -3 \cdot (n+4) \cdot \left(\frac{1}{2}\right)^{n} + \left(\frac{2}{3}\right)^{n+1} \cdot 19$$

$$x_n = -3 \cdot (n+3) \cdot \left(\frac{1}{2}\right)^{n-1} + 19 \cdot \left(\frac{2}{3}\right)^{n}$$

Un şir s. n. crescător dacă ...

mărginit ...

(4) Fie $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$, $x_0 = 2$

Inducţie $\Rightarrow$ $x_n > 0$, $(\forall)\ n$

$$x_{n+1} = \frac{1}{2} x_n + \frac{1}{x_n}$$

Monotonie şi mărginire

$$x_{n+1} - x_n = \frac{1}{x_n} - \frac{1}{2} x_n > 0 \iff$$

$$\iff 2 > x_n^2 \iff x_n < \sqrt{2}$$

$x_0 > \sqrt{2}$

O fi şirul $\searrow$

$$x_{n+1} - \sqrt{2} = \frac{1}{2} x_n + \frac{1}{x_n} - \sqrt{2}$$

$$= \frac{x_n^2 - 2\sqrt{2} x_n + 2}{2 x_n}$$

$$= \frac{(x_n - \sqrt{2})^2}{2 x_n} \geq 0 \Rightarrow$$

$\Rightarrow x_n \geq \sqrt{2}$, $(\forall)$ n

Deci $x_n \searrow$ și mărg. (în $(0, 2)$)

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(5) $a_1 = 1$

$a_{n+1} (a_n + 1) = a_n + 2$

Inducție $a_n > 0$, $(\forall) n$

$$a_{n+1} = \frac{a_n + 2}{a_n + 1} = 1 + \frac{1}{a_n + 1}$$

$$1, \ \frac{3}{2}, \ 1 + \frac{1}{\frac{3}{2} + 1} = 1 + \frac{2}{5} = \frac{7}{5}$$

$$1 < \frac{3}{2} > \frac{7}{5}$$

$$1 + \frac{1}{\frac{7}{5}+1} = 1 + \frac{5}{12} = \frac{17}{12}$$

$$\frac{7}{5} < \frac{17}{12} \qquad\qquad \frac{3}{2} > \frac{17}{12}$$

$$1 < \frac{3}{2} > \frac{7}{5} < \frac{17}{12}$$

$$a_{n+1} = 1 + \frac{1}{a_n+1}$$

$$= 1 + \frac{1}{1+\frac{1}{1+a_{n-1}}}$$

$$a_{n+1} = 1 + \frac{1}{1+\frac{1}{1+a_{n-3}}}$$

$$a_{n-1} > a_{n-3} \Rightarrow a_{n+1} > a_{n-1}$$

$$a_{n-1} < a_{n-3} \Rightarrow a_{n+1} < a_{n-1}$$

Deci $(a_{2n})_n \nearrow \quad (a_{2n+1})_n \searrow$

Enunț: Arătați că (a_{2n}) și

(a_{2n+1}) au

monotonii

diferite

și $\left(\left|a_n-\sqrt{2}\right|\right)_n$ e monoton.

$$\left(a_{n+1}-\sqrt{2}\right) = 1+\frac{1}{a_n+1}-\sqrt{2}$$

$$= \frac{a_n+2-\sqrt{2}\,a_n-\sqrt{2}}{a_n+1}$$

$$= \frac{a_n-\sqrt{2}-\sqrt{2}\left(a_n-\sqrt{2}\right)}{a_n+1}$$

$$= \frac{\left(a_n-\sqrt{2}\right)\cdot\left(1-\sqrt{2}\right)}{a_n+1}$$

$$\left|a_{n+1}-\sqrt{2}\right| \le \left|a_n-\sqrt{2}\right|\cdot\left(1-\sqrt{2}\right) \le \left|a_n-\sqrt{2}\right|$$

$(a_n+1\geq 1)$

⑥ Să se arate că orice progresie aritmetică de nr. naturale conține termenii unei progr. geometrice

$$a_1,\ a_1+r,\ a_1+2r,\ \dots,\ \underbrace{a_1(1+r)}_{\text{seamănă cu progr. geom.}},\ \dots$$

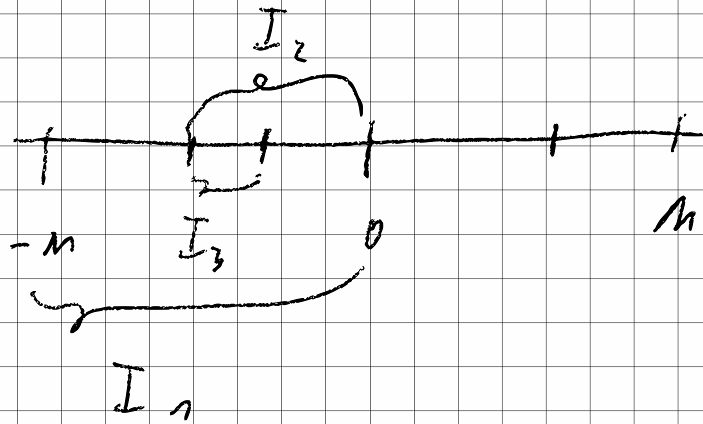
$a_1 + a_1 r$

$$a_1(1+r)^2 = a_1 + r(2a_1 + r a_1)$$

$$a_1(1+r)^3 = a_1 + r(3a_1 + 3r a_1 + r^2 a_1)$$

$\vdots$

⑦ Fie $(x_n)_n$ un șir mărginit cu termeni distincți 2 câte 2. Arătați că există pe dreapta reală un interval oricât de mic care să conțină o infinitate de termeni. E mărginirea o ipoteză esențială?



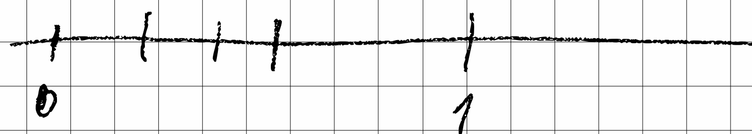
$$I_0 = [a_0, b_0] = [-n, n]$$

$$I_1 = [a_1, b_1] = \begin{cases} \left[a_0, \frac{a_0+b_0}{2}\right] & \text{dacă conține o inf.} \\ \left[\frac{a_0+b_0}{2}, b_0\right] & \text{dacă cont. o infinitate} \end{cases}$$

$$l(I_1) = \frac{l(I_2)}{2}$$

(8) Există vreo progresie geometrică de nr. reale a.î., pentru orice nr. natural nenul n, în intervalul $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ să existe termeni ai progresiei?

Dar progr. aritm.



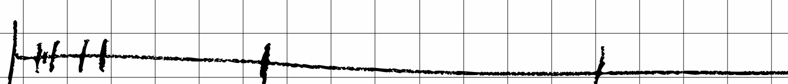
Putem presupune $x_0 > 0$

$q > 0$

Dacă nu, schimbăm $x_n \longrightarrow |x_n|$

Dacă $q \geq 1$ X

Rămâne $q < 1$ $\Rightarrow$ $(x_n)_n \searrow$



Dt. n suf. de mic, $(\exists)$ termeni ai

progr. şi în $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ şi în $\left[\frac{1}{n+2}, \frac{1}{n+1}\right]$

Fie $x_n \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$

Vrem $x_{n+1} < \frac{1}{n+2}$

$$q \cdot x_n < \frac{q}{n} \overset{(\text{Vrem})}{\leq} \frac{1}{n+2} \quad q \leq \frac{n}{n+2}$$

$$(n+2) \cdot q \leq n$$

$$2 \cdot q \leq n(1-q)$$

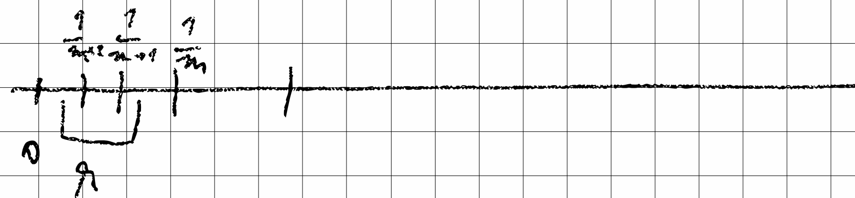
$$n \geq \frac{2 \cdot q}{1-q} > 0$$

Aleg $n = \left[\frac{2 \cdot q}{1-q}\right] + 1$ ✓

Pt. progr. aritmetică

Dacă $r = 0$, clar

$x_n = x_0 + r \cdot n$



Dacă $|r| > \frac{1}{n} - \frac{1}{n+2}$, atunci

~~(∃)~~ termeni ai progr. și în $\left[\frac{1}{n+1}, \frac{1}{n+2}\right]$ și în $\left[\frac{1}{n+1}, \frac{1}{n}\right]$

$\Downarrow$ $\Downarrow$

Presupunem x_{n+k} x_n

$$|x_n - x_{n+k}| < \frac{1}{n} - \frac{1}{n+2} < |r|$$

$$\frac{1}{n+1} < x_n < \frac{1}{n}$$

$$-\frac{1}{n+2} > -x_{n+k} > -\frac{1}{n+1}$$

$$\frac{1}{n} - \frac{1}{n+2} > x_n - x_{n+k} > 0$$

$$|h| \cdot |r| < |r|$$

Fals

Vrem $|r| \geq \frac{1}{n} - \frac{1}{m+2}$

$$|r| \geq \frac{2}{m(m+2)}$$

$$m(m+2) \geq \frac{2}{|r|}$$

Dacă $m^2 \geq \frac{2}{|r|}$ OK

$$m \geq \left\lceil \sqrt{\frac{2}{|r|}} \right\rceil + 1$$

⑨ Să se arate că, într-o progresie aritmetică de nr. naturale, există o infinitate de termeni cu aceeași sumă a cifrelor

Fie $s(x)$ = suma cifrelor lui x

$$x_n = x_0 + n \cdot r$$

Fie $k \in \mathbb{N}$ a.î.

$$10^k > x_0$$

$$x_0 = \overline{x_0^1 x_0^2 \dots x_0^m}$$

$$r = \overline{r^1 r^2 \dots r^p}$$

$$x_{10^k} = x_0 + 10^k \cdot r$$

$$k \geq m$$

$$x_{10^k} = \overline{r^1 r^2 \dots r^p \underbrace{0\,0 \dots 0}_{k-m} x_0^1 x_0^2 \dots x_0^m}$$

$$s(x_{10^k}) = s(r) + s(x_0) = \text{const.}, \ (\forall)\ k \geq m$$

(10) Fie $a, b, c \in \mathbb{N}$ distincte 2 câte 2

$$(a, b, c) = 1$$

Dacă $\sqrt{a}, \sqrt{b}, \sqrt{c}$ sunt termeni ai unei progresii aritmetice (nu neapărat consecutivi), atunci $\sqrt{a}, \sqrt{b}, \sqrt{c} \in \mathbb{N}$

Sol:

Fie $\sqrt{b} = \sqrt{a} + m \cdot r$

$$\sqrt{c} = \sqrt{a} + n \cdot r$$

a, b, c distincte $\Rightarrow r, m, n$ nenule

$$\frac{\sqrt{b}-\sqrt{a}}{\sqrt{c}-\sqrt{a}} = \frac{m}{n} = q \in \mathbb{Q}_+^*$$

$b \neq c \Rightarrow q \neq 1$

$$\sqrt{b}-\sqrt{a} = q\cdot(\sqrt{c}-\sqrt{a})$$

$$\sqrt{b} + \underbrace{(q-1)}_{p}\sqrt{a} - q\cdot\sqrt{c} = 0$$

$p \in \mathbb{Q}^*$

$$\sqrt{b} + p\sqrt{a} - q\sqrt{c} = 0$$

$$\sqrt{b} = -p\sqrt{a} + q\sqrt{c} \quad \Big|^2$$

$$b = p^2 a + q^2 c - p\cdot q\cdot\sqrt{ac} \Rightarrow$$

$$\Rightarrow \sqrt{ac} \in \mathbb{Q} \Rightarrow ac = k^2$$

Analog $\sqrt{ab}, \sqrt{bc} \in \mathbb{Q} \Rightarrow$

$\Rightarrow ab = l^2$

Pp. că $\sqrt{a} \notin \mathbb{N} \Rightarrow (\exists)\ p$ prim a.î.

exponentul lui p în a $\quad v_p(a) =$ impar $\Big/ \Rightarrow$

$ac = k^2$

$\Rightarrow p \mid c$

$a b = d^2 \Rightarrow p \mid b$

$p \mid (a, b, c) = 1$

Fals.

$\blacksquare$